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LINEAR SEQUENTIAL q-DIFFERENCE EQUATIONS OF FRACTIONAL ORDER [1]

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Abstract

We prove the existence and uniqueness of solutions of sequential linear fractional q-difference equations where the fractional derivative is the Riemann–Liouville fractional q-derivative. A fundamental set of solutions is derived for the homogenous linear sequential fractional difference equations with constant coefficients and a general solution for the corresponding non homogenous equation is constructed by using the q-Laplace transform method. These results extend the results of Annaby et al. in [3] and Kilbas et al. in [20].

Mathematics Subject Classification: 26A33, 34A12, 39A13

Key Words and Phrases: q-integrals and q-derivatives of fractional order, existence and uniqueness theorems, q, α Wronskian, fundamental set of solutions, q-Laplace transform method

1. Introduction

In the following q is a positive number, $0 < q < 1$, and by the word basic we mean a q-analog. For $n \in \{1, 2, \ldots\}$, $\beta, a, a_1, \ldots, a_k, \in \mathbb{C}$, and $\nu \in \mathbb{R}$ we define the following functions and notations

$$\begin{aligned}(a;q)_0 &:= 1,\ (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),\ (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),\\ (a;q)_\nu &:= \frac{(a;q)_\infty}{(aq^\nu;q)_\infty},\ \begin{bmatrix}\beta\\0\end{bmatrix}_q := 1,\ \ \begin{bmatrix}\beta\\n\end{bmatrix}_q := \frac{(q^{\beta-n+1};q)_n}{(q;q)_n}.\end{aligned}$$

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The q-gamma and the q-beta functions, cf. [11, 17], are defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z},\ z\in\mathbb{C},\ z\notin\{0,-1,-2,\ldots\}, \tag{1.1}$$

$$B_q(a,b) := \int_0^1 x^{a-1}(qx;q)_{b-1}\,d_qx, \quad a,b>0. \tag{1.2}$$

It is known that $B_q(a,b)=\dfrac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}$, cf. [8, p.494]. Let f be a function defined on a q-geometric set A, I.e. $qx\in A$ for all $x\in A$. The q-difference operator is defined by

$$D_qf(x) := \begin{cases} \dfrac{f(x)-f(qx)}{x-qx}, & x\in A-\{0\}, \\ \lim\limits_{\substack{n\to\infty \\ x\in A-\{0\}}} \dfrac{f(xq^n)-f(0)}{xq^n}, & x=0, \end{cases} \tag{1.3}$$

provided that the limit exists and does not depend on x. The Jackson q-integration, cf. [18], is

$$\int_0^a f(t)\,d_qt := a(1-q)\sum_{n=0}^\infty q^nf(aq^n),\ \int_a^b f(t)\,d_qt := \Big(\int_0^b - \int_0^a\Big)f(t)\,d_qt, \tag{1.4}$$

where $a,b\in A$, provided that the series converge. If $0\in A$, f is called q-regular at zero if $\lim\limits_{n\to\infty} f(xq^n)=f(0)$ for every $x\in A$, $x\neq 0$. The q-integration by part rule is

$$\int_a^b u(qt)D_qv(t)\,d_qt = u(b)v(b)-u(a)v(a)+\int_a^b D_qu(t)v(t)\,d_qt, \tag{1.5}$$

provided that u and v are q-regular at zero functions. By $L_q^1(0,a)$, $a>0$, we mean the Banach space of all functions defined on $(0,a]$ such that

$$\|f\| := \int_0^a |f(t)|\,d_qt < \infty. \tag{1.6}$$

Let $\mathcal{L}_q^1(0,a)$ denote the space of all functions f defined on $(0,a]$ such that $f\in L_q^1(0,x)$ for all $x\in(qa,a]$. The space $\mathcal{AC}_q[0,a]$ is the space of all functions f defined on $[0,a]$ such that f is q-regular at zero and

$$\sum_{j=0}^\infty \big|f(tq^j)-f(tq^{j+1})\big| < \infty, \quad t\in(qa,a]. \tag{1.7}$$

Let $\mathcal{A}C_q^{(k)}[0,a]$, $k \in \mathbb{Z}^+$, be the space of all functions f defined on $[0,a]$ such that f, $D_q f, \dots, D_q^{k-1} f$ are q-regular at zero and $D_q^{k-1} f \in \mathcal{A}C_q[0,a]$. When $k=1$ we simply write $\mathcal{A}C_q[0,a]$ for $\mathcal{A}C_q^{(1)}[0,a]$.

The Riemann–Liouville (R-L) fractional q-integral operator is introduced in [6] by Al-Salam through

$$I_q^\alpha f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t)\, d_q t, \quad \alpha \notin \{-1,-2,\dots\}. \tag{1.8}$$

Using (1.4), (1.8) reduces to

$$I_q^\alpha f(x) = x^\alpha (1-q)^\alpha \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n), \tag{1.9}$$

which is valid for all α. This R-L fractional q-integral is an extension of the basic Cauchy formula,

$$\begin{aligned} I_{q,a}^n f(x) &:= \int_a^x \int_a^{x_{n-1}} \dots \int_a^{x_1} f(t)\, d_q t d_q x_1 \dots d_q x_{n-1} \\ &= \frac{x^{n-1}}{\Gamma_q(n)} \int_a^x (qt/x;q)_{n-1} f(t)\, d_q t. \end{aligned}$$

introduced by Al Salam in [7]. This basic R-L fractional integral is also given later by Agarwal in [5]. In [9] the fractional q-derivative of order α, $\alpha > 0$ is defined by

$$D_q^\alpha f(x) := \phi(x) = D_q^k I_q^{k-\alpha} f(x), \quad k = [\alpha]+1, \tag{1.10}$$

provided that

$$f(x) \in \mathcal{L}_q^1[0,a], \quad I_q^{k-\alpha} f(x) \in \mathcal{A}C_q^{(k)}[0,a]. \tag{1.11}$$

The following semi-group property is established by R.P. Agarwal in [5]:

$$I_q^\alpha I_q^\beta f(x) = I_q^\beta I_q^\alpha f(x) = I_q^{\alpha+\beta} f(x), \quad \alpha, \beta \geqslant 0, \tag{1.12}$$

The following two properties which are proved in [9] will be needed in the sequel:

1. If $f \in \mathcal{L}_q^1[0,a]$, then
$$D_q^\alpha I_q^\alpha f(x) = f(x); \quad \alpha > 0, \; x \in (0,a]. \tag{1.13}$$
2. If $f \in \mathcal{L}_q^1[0,a]$ and $I_q^{1-\alpha} f \in \mathcal{A}C_q[0,a]$, then
$$I_q^\alpha D_q^\alpha f(x) = f(x) - I_q^{1-\alpha} f(x)\Big|_{x=0} \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}, \quad x \in (0,a]. \tag{1.14}$$

Two q-analogs of the exponential function are introduced by Jackson [16] and defined by

$$E_q(z) := (z;q)_\infty,\ z \in \mathbb{C}, \quad \text{and} \quad e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n},\ |z| < 1. \tag{1.15}$$

A q-analog of the Mittag–Leffler function is introduced in [9] and defined by

$$e_{\nu,\mu}(z;q) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\nu n + \mu)}, \quad |z| < (1-q)^{-\nu}, \tag{1.16}$$

where $\nu > 0,\ \mu \in \mathbb{C}$. For further studies on the Mittag-Leffler functions and the R-L fractional operator, see for example [10, 12, 13, 15, 19, 23].

From now on, α will denote a positive number which is less than one.

2. Sequential linear q-difference equations of fractional order

In this section we prove the existence and uniqueness theorem for first order system of R-L fractional q-derivative of order α. Based on this result, the existence and uniqueness theorem for solutions of linear sequential q-difference equation is introduced. We also introduce and prove some essential properties of the q, α Wronskian, see [20, Chapter 7] for the definition of the α Wronskian.

Definition 2.1. Let $n \in \mathbb{N}$. We shall call linear sequential fractional q-difference equation of order $n\alpha$ the equation of the form

$$\sum_{k=0}^{n} b_k(x) \mathcal{D}_q^{k\alpha} y(x) = g(x), \quad 0 < x < a, \tag{2.1}$$

where $b_k,\ k = 0, 1, \ldots, n$, and g are given real functions, $b_n(x) \neq 0$ for all $x \in (0, a)$, and $\mathcal{D}_q^{k\alpha} y$ is the sequential fractional q-derivatives corresponding to the R-L fractional operator defined by:

$$\mathcal{D}_q^{\alpha} y := D_q^{\alpha} y, \ \ \mathcal{D}_q^{k\alpha} y := D_q^{\alpha} \mathcal{D}_q^{(k-1)\alpha} y, \quad k = 2, 3, \ldots.$$

If for all $x \in [0, a]$, $b_n(x) \neq 0$, equation (2.1) may be expressed in its normal form as follows:

$$\left(L_{q,n\alpha} y\right)(x) := \mathcal{D}_q^{n\alpha} y(x) + \sum_{k=0}^{n-1} a_k(x) \mathcal{D}_q^{k\alpha} y(x) = f(x). \tag{2.2}$$

The following lemma which is introduced in [4] is essential in our investigations.

LEMMA 2.1. *Let I and J be intervals containing zero, such that $J \subseteq I$. Let f_n, f be functions defined in I, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} f_n(t) = f(t)$, for all $t \in I$, and $(f_n)_n$ converges uniformly to f on J. Then*

$$\lim_{n\to\infty} \int_0^x f_n(t)\, d_qt = \int_0^x f(t)\, d_qt, \qquad \text{for all } x \in I.$$

THEOREM 2.1. *Let $f_i(x, y_1, \ldots, y_n)$ be functions defined for $x \in (0, a]$, $a > 0$, and y_i in domains $G_i \subseteq \mathbb{C}$, $i = 1, 2, \ldots, n$, satisfying the following conditions:*

(i) There is a positive constant A such that, for $x \in (0, a]$ and $y_i, \tilde{y_i} \in G_i$, $1 \leqslant i \leqslant n$, the following Lipschiz' condition is fulfilled

$$|f_i(x, y_1, \ldots, y_n) - f_i(x, \tilde{y_1}, \ldots, \tilde{y_n})| \leqslant A\Big(|y_1 - \tilde{y_1}| + \ldots + |y_n - \tilde{y_n}|\Big).$$

(ii) There exists $M > 0$ such that

$$|f_i(x, y_1, \ldots, y_n)| \leqslant Mx^{\alpha-1},\ y_i \in G_i,\ i = 1, \ldots, n,\ x \in (0, a]. \tag{2.3}$$

Let K be a constant that satisfies

$$K \geqslant \frac{Ma^{2\alpha-1}\Gamma_q(\alpha)}{\Gamma_q(2\alpha)}, \tag{2.4}$$

and $D_i(a, K) \subset G_i$ be the set of points $y_i \in G_i$ satisfying the relation

$$|y_i - b_i \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}| < K, \quad \text{for all } x \in (0, a].$$

Then, there exists $h \in (0, a]$ such that the initial value problem

$$D_q^\alpha y_i(x) = f_i(x, y_1(x), \ldots, y_n(x)), \quad i = 1, 2, \ldots, n, \tag{2.5}$$

$$I_q^{1-\alpha} y_i(x)|_{x=0} = b_i, \quad b_j \in \mathbb{R}, \qquad i = 1, 2, \ldots, n, \tag{2.6}$$

has a unique solution $\{\phi_i(\cdot)\}_{i=1}^n$ valid in $(0, h]$. Moreover, the functions $\{\phi_i\}_{i=1}^n$ are 0^+-singular of order α, that is

$$\lim_{x\to 0^+} x^{1-\alpha}\phi_i(x) < \infty,\ i = 1, 2, \ldots, n.$$

P r o o f. *Existence.* Define the sequences $\{\phi_{i,m}(x)\}_{i,m=1}^{\infty}$, $x \in (0,a]$ by $\phi_{i,1}(x) = \dfrac{b_i\, x^{\alpha-1}}{\Gamma_q(\alpha)}$, and

$$\phi_{i,m}(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\left(b_i + \int_0^x (qt/x;q)_{\alpha-1} f_i\left(t, \phi_{1,m-1}(t), \ldots, \phi_{n,m-1}(t)\right)\, d_qt\right), \tag{2.7}$$

$m \geqslant 2$. We will show that $\lim\limits_{m\to\infty} \phi_{i,m}(x)$ exists and gives the required solution $\{\phi_i(\cdot)\}_{i=1}^n$ of the initial value problem (2.5)–(2.6). We prove the existence in four steps.

i. We prove by induction on m that

$$\phi_{i,m}(x) \in D_i(a,k), \quad m \in \mathbb{Z}^+, \quad x \in (0,a], \quad i = 1,2,\ldots,n. \tag{2.8}$$

Clearly, $\phi_{i,1}(x) \in D_i(a,K)$ for $i = 1,\ldots,n$ and $x \in (0,a]$. If we assume that $\phi_{i,m}(x) \in D_i(a,k)$ for $i = 1,\ldots,n$ and $x \in (0,a]$, then by (2.3) we obtain

$$|f_i(x, \phi_{1,m}(x), \ldots, \phi_{n,m}(x))| \leqslant Mx^{\alpha-1}, \quad x \in (0,a].$$

Thus

$$\begin{aligned}\left|\phi_{i,m+1}(x) - \frac{b_i}{\Gamma_q(\alpha)}x^{\alpha-1}\right| &\leqslant \frac{Mx^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{\alpha-1}\, d_qt \\ &= \frac{Mx^{2\alpha-1}}{\Gamma_q(\alpha)} B_q(\alpha,\alpha) = \frac{Mx^{2\alpha-1}\Gamma_q(\alpha)}{\Gamma_q(2\alpha)} \\ &\leqslant \frac{Ma^{2\alpha-1}\Gamma_q(\alpha)}{\Gamma_q(2\alpha)} \leqslant K. \end{aligned} \tag{2.9}$$

So, $\phi_{i,m+1}(x) \in D_i(a,K)$, $x \in (0,a]$. This completes the induction steps and proves (2.8).

ii. We prove that $\phi_{i,m}$ is 0^+-singular of order α for all $m \in \mathbb{N}$, $i = 1,2,\ldots,n$. From (2.8) we conclude that

$$\left|\phi_{i,m}(x) - \frac{b_i}{\Gamma_q(\alpha)}x^{\alpha-1}\right| \leqslant K, \quad m \in \mathbb{Z}^+,\ x \in (0,a].$$

Consequently,

$$\left|x^{1-\alpha}\phi_{i,m}(x) - \frac{b_i}{\Gamma_q(\alpha)}\right| \leqslant Kx^{1-\alpha}, \quad m \in \mathbb{Z}^+, \quad x \in (0,a].$$

Therefore $\lim_{x\to 0^+} x^{1-\alpha}\phi_{i,m}$ exists for all $m \in \mathbb{N}$, $i = 1,2,\ldots,n$.

iii. We prove by induction on m that

$$|\phi_{i,m+1}(x)-\phi_{i,m}(x)| \leqslant \frac{MB^{m-1}x^{m\alpha}}{\Gamma_q(m\alpha+1)},\ B:=An,\ m\in\mathbb{Z}^+,\ x\in(0,a]. \tag{2.10}$$

From (2.9), inequality (2.10) is true at $m=1$. Assume that (2.10) is true at $m=k$. Hence for $x\in(0,a]$, we have

$$\begin{aligned}
&|\phi_{i,k+2}(x)-\phi_{i,k+1}(x)| \leqslant \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\\
&\times |f_i(t,\phi_{1,k+1}(t),\dots,\phi_{n,k+1}(t)) - f_i(t,\phi_{1,k}(t),\dots,\phi_{n,k}(t))|\, d_qt\\
&\leqslant \frac{Ax^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}\sum_{j=1}^n |\phi_{j,k+1}(t)-\phi_{j,k}(t)|\, d_qt\\
&\leqslant \frac{MB^kx^{\alpha-1}}{\Gamma_q(\alpha)\Gamma_q(k\alpha+1)}\int_0^x (qt/x;q)_{\alpha-1}t^{k\alpha}\, d_qt = \frac{B^kM}{\Gamma_q((k+1)\alpha+1)}x^{(k+1)\alpha}.
\end{aligned}$$

That is (2.10) is true at $m=k+1$ and hence it is true for all $m\geqslant 1$.

iv. We prove that $\lim_{m\to\infty}\phi_{i,m}(x)$ exists for $i=1,2,\dots,n$, $x\in(0,a]$ such that $\{\phi_i(\cdot)\}_{i=1}^n$,

$$\phi_i(x) := \lim_{m\to\infty}\phi_{i,m}(x),\quad i=1,2,\dots,n,\quad x\in(0,a],$$

defines a solution of (2.5), (2.6). Consider the infinite series

$$\phi_{i,1}(x)+\sum_{m=1}^{\infty}\phi_{i,m+1}(x)-\phi_{i,m}(x). \tag{2.11}$$

From (2.10) we obtain

$$\sum_{m=1}^{\infty}|\phi_{i,m+1}(x)-\phi_{i,m}(x)| \leqslant \frac{M}{B}\sum_{m=1}^{\infty}\frac{(Bx^\alpha)^m}{\Gamma_q(m\alpha+1)} \leqslant \frac{M}{B}e_{\alpha,\alpha}(Bx^\alpha;q).$$

Set $h:=\min\left\{a,\frac{1}{B^{1/\alpha}(1-q)}\right\}$. Since $e_{\alpha,\alpha}(Bx^\alpha;q)$ is defined only for $|x|\leqslant h$, then the series in (2.11) is uniformly convergent on $(0,h]$ to a function ϕ_i, and $\phi_i(x)=\lim\limits_{m\to\infty}\phi_{i,m}(x)$. Since $\phi_{i,m}(x)$ is 0^+-singular of order α for all $m\in\mathbb{N}$, $i=1,\dots,n$, then so is $\phi_i(x)$. Also $\phi_{i,m}(x)\in D_i(a,k)$ implies that $\phi_i(x)\in D_i(a,k)$, $x\in(0,h]$. The uniform convergence of the sequences $\{\phi_{i,m}(x)\}$ on $(0,h]$ allows us to let $m\to\infty$ in the relationship (2.7), which gives

$$\phi_i(x)=\frac{b_i}{\Gamma_q(\alpha)}x^{\alpha-1}+\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}f_i\left(t,\phi_1(t),\dots,\phi_n(t)\right)\, d_qt,$$

$x \in (0, a]$. Consequently applying (1.12) yields

$$\begin{aligned} I_q^{1-\alpha}\phi_i(x) &= b_i + I_q^{1-\alpha} I_q^{\alpha} f_i(x, y_1(x), \ldots, y_n(x)) \\ &= b_i + I_q f_i(x, y_1(x), \ldots, y_n(x)). \end{aligned}$$

Since $f_i(x, y_1(x), \ldots, y_n(x))$, $i = 1, 2, \ldots, n$, are bounded on $(0, h]$, then $I_q^{1-\alpha}\phi_i(0) = b_i$, $i = 1, 2, \ldots, n$, i.e. $\{\phi_i\}_{i=1}^n$ satisfies the initial conditions (2.6).

Uniqueness. To prove uniqueness, we assume that $\{\psi_i\}_{i=1}^n$ is another solution of (2.5)–(2.6) valid in an interval $(0, b]$, $b \leqslant h$. For $i = 1, 2, \ldots, n$, and $x \in (0, h]$ set

$$\begin{aligned} \chi_i(x) &:= \phi_i(x) - \psi_i(x), \\ g_i(x) &:= f_i\left(x, \phi_1(x), \ldots, \phi_n(x)\right) - f_i\left(x, \psi_1(x), \ldots, \psi_n(x)\right) \end{aligned}$$

Hence $D_q^{\alpha}\chi_i(x) = g_i(x)$, $I_q^{1-\alpha}\chi_i(0) = 0$, $i = 1, \ldots, n$. By (1.14) we obtain

$$\chi_i(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x; q)_{\alpha-1} g_i(t)\, d_q t, \; i = 1, \cdots, n. \tag{2.12}$$

Now χ_i is 0^+-singular of order α for $i = 1, 2, \ldots, n$. Then for each $x \in (qb, b)$ there exists a constant $C_x > 0$ such that

$$t^{1-\alpha}|\chi_i(t)| \leqslant \frac{C_x}{\Gamma_q(\alpha)}, \; t \in \{xq^m, \; m \in \mathbb{N}\}. \tag{2.13}$$

Fix $x \in (qh, h]$ and $t \in \{xq^m, \; m \in \mathbb{N}\}$. Hence from (2.12) and (2.13) we obtain

$$\begin{aligned} |\chi_i(t)| &\leqslant \frac{At^{\alpha-1}}{\Gamma_q(\alpha)} \sum_{i=1}^{n} \int_0^t (qu/t; q)_{\alpha-1} \chi_i(u)\, d_q u \\ &\leqslant \frac{AnC_x t^{2\alpha-1}}{\Gamma_q^2(\alpha)} B_q(\alpha, \alpha) = \frac{BC_x}{\Gamma_q(2\alpha)} t^{2\alpha-1} \end{aligned}$$

Repeating the previous process k times we obtain

$$|\chi_i(t)| \leqslant C_x \frac{B^{k+1} t^{\alpha k+\alpha-1}}{\Gamma_q(\alpha k+\alpha)}, \;\; k \in \mathbb{Z}^+, \;\; i = 1, 2, \ldots, n. \tag{2.14}$$

Since $\dfrac{B^k t^{\alpha k}}{\Gamma_q(\alpha k+\alpha)}$ is the general term of the series of $e_{\alpha,\alpha}(\beta t^{\alpha})$, $t < a$, then $\lim_{k\to\infty} \dfrac{B^k t^{\alpha k}}{\Gamma_q(\alpha k+\alpha)} = 0$. Therefore $\chi_i(t) = 0$, $t \in \{xq^m, m \in \mathbb{N}\}$, $x \in (qb, b]$. That is $\chi(x) = 0$ for all $x \in (0, b]$, proving the uniqueness. ∎

THEOREM 2.2 . *Assume that all conditions of Theorem* 2.1 *above are satisfied with* $G_i = \mathbb{C}$ *for all* i; $i = 1, \ldots, n$. *Then problem* (2.5)–(2.6) *has a unique solution valid at least in* I^*, $I^* := (0, a] \bigcap \left(0, \frac{1}{(An)^{\frac{1}{\alpha}}(1-q)}\right)$.

P r o o f. We prove this theorem by proving the existence and uniqueness on any subinterval $(0, h] \subseteq I^*$, $h > 0$. Similarly to the proof of Theorem 2.1 above, we can find a constant $\gamma \leqslant h$ such that $\phi_{i,m}$ converges uniformly to ϕ on $(0, \gamma]$, where $\phi_{i,m}$ are defined in (2.7). In addition to this it is not hard to see that $\phi_{i,m}$ converges to ϕ_i pointwise on $(0, h]$. Using Lemma 2.1 it can be shown that the solution $\{\phi_i\}_{i=1}^n$ could be extended throughout $(0, h]$. ■

THEOREM 2.3. *Let* $a_j(x)$, $j = 0, 1, \ldots, n-1$, *and* $f(x)$ *be* 0^+*-singular functions of order* α *defined on* $(0, a]$. *Then there exists* $0 < h \leqslant a$ *such that the equation*

$$L_{n\alpha,q} y(x) = f(x) \tag{2.15}$$

has a unique solution valid in $(0, h]$ *satisfying for* $k = 0, 1, \ldots, n-1$,

$$I_q^{1-\alpha} \mathcal{D}_q^{k\alpha} y(0) = b_k, \quad b_k \in \mathbb{R}, \tag{2.16}$$

or equivalently,

$$\lim_{x \to 0^+} x^{1-\alpha} \mathcal{D}_q^{k\alpha} y(x) = \frac{b_k}{\Gamma_q(\alpha)}. \tag{2.17}$$

P r o o f. From Theorem 2.1, there exists $h > 0$ such that the first order system

$$\begin{aligned} D_q^\alpha y_i &= y_{i+1}, \quad i = 1, 2, \ldots, n-1, \\ D_q^\alpha y_n &= -a_0 y_1 - a_1 y_2 - \ldots - b_{n-1} y_n + f(x) \end{aligned} \tag{2.18}$$

has a unique solution valid in $(0, h]$. But $\{y_j\}_{j=1}^n$ is a solution of (2.18) with the initial conditions (2.6) if and only if y_1 is a solution of (2.15) with the initial conditions (2.16) or (2.17). ■

The following two propositions follow at once from Theorem 2.3.

PROPOSITION 2.1. *Let* a_j, $j = 1, 2, \ldots, n$, *be* 0^+*-singular of order* α *functions defined on* $(0, a]$. *Then the homogeneous fractional q-difference equation* (2.2) *with the initial conditions*

$$\lim_{x \to 0^+} x^{1-\alpha} \mathcal{D}_q^{j\alpha} y(x) = 0, \quad \text{or} \quad I_q^{1-\alpha} \mathcal{D}_q^{j\alpha} y(0) = 0, \quad j = 0, 1, \ldots, n-1,$$

has only the trivial continuous solution $y(x) = 0$.

Proposition 2.2. *Any linear combination of solutions of the homogeneous equation*

$$L_{q,n\alpha}y(x) = 0 \tag{2.19}$$

is also a solution of this equation.

Definition 2.2. We call q, α Wronskian of n functions u_j, $j = 1, 2, \ldots, n$, having fractional sequential q-derivative up to order $(n-1)\alpha$ in $(0, a]$, the following determinant

$$|W_{q,\alpha}(u_1, \ldots, u_n)(x)| =$$

$$\begin{vmatrix} u_1(x) & u_2(x) & \ldots & u_n(x) \\ \mathcal{D}_q^{\alpha} u_1(x) & \mathcal{D}_q^{\alpha} u_2(x) & \ldots & \mathcal{D}_q^{\alpha} u_n(x) \\ \vdots & & & \vdots \\ \mathcal{D}_q^{(n-1)\alpha} u_1(x) & \mathcal{D}_q^{(n-1)\alpha} u_2(x) & \ldots & \mathcal{D}_q^{(n-1)\alpha} u_n(x) \end{vmatrix}.$$

To simplify the notation, this will be represented by $|W_{q,\alpha}(x)|$. We shall use $W_{q,\alpha}$ for the corresponding q, α Wronskian matrix.

Proposition 2.3. *Let* $\{u_j(x)\}_{j=1}^n$ *be a family of functions which admit fractional sequential* q*-derivatives up to order* $(n-1)\alpha$ *in* $(0, b]$*, satisfying for* $j = 1, 2, \ldots, n$ *and* $k = 0, \ldots, n-1$*, the condition*

$$\lim_{x \to 0^+} x^{1-\alpha} \mathcal{D}_q^{k\alpha} u_j(x) < \infty. \tag{2.20}$$

If the functions $\left\{x^{1-\alpha} u_j(x)\right\}_{j=1}^n$ *are linearly dependent in* $[0, a]$*, then*

$$x^{n-n\alpha} |W_{q,\alpha}(x)| = 0, \quad \textit{for all} \quad x \in [0, a].$$

P r o o f. Since $\left\{x^{1-\alpha} u_j(x)\right\}_{j=1}^n$ are linearly dependent in $[0, a]$, then there exist n constants $\{c_j\}_{j=1}^n$, not all zeros, such that for all $x \in [0, a]$, $\sum_{j=1}^n c_j x^{1-\alpha} u_j(x) = 0$. Therefore for all $x \in (0, a]$ we get

$$\sum_{k=1}^n c_j u_j(x) = 0. \tag{2.21}$$

Successive applications of the sequential q-derivative $\mathcal{D}_q^{k\alpha}$, $k = 1, \ldots, n-1$, to (2.21) lead to the following relation:

$$W_{q,\alpha}(x) \widetilde{C} = \widetilde{0}, \quad \widetilde{C} = (c_1, c_2, \ldots, c_n)^{\top} \neq \widetilde{0}, \tag{2.22}$$

and $\widetilde{0}$ is the zero $n \times 1$ matrix. Consequently, $|W_{q,\alpha}(x)| = 0$ for all $x \in (0, a]$. In addition, by (2.20), we can also conclude that

$$\lim_{x\to 0^+} x^{1-\alpha} W_{q,\alpha}(x)\widetilde{C} = \widetilde{0},$$

and hence $\lim_{x\to 0^+} x^{n-n\alpha}|W_{q,\alpha}(x)| = 0$, which completes the proof. ∎

PROPOSITION 2.4. *Let* $\{u_j(x)\}_{j=1}^n$ *be solutions of the equation* (2.19) *satisfying the initial conditions* (2.16) *or* (2.17). *Then* $\{u_j(x)\}_{j=1}^n$ *are linearly independent if and only if*

$$\lim_{x\to 0^+} x^{n-n\alpha} \left| W_{q,\alpha}\left(u_1,\dots,u_n\right)\right| \neq 0. \tag{2.23}$$

P r o o f. The proof of the sufficient part follows from Proposition 2.3 by 'reductio ad absurdum'. To prove the necessary part, we suppose on the contrary that $\lim_{x\to 0^+} x^{n-n\alpha}|W_{q,\alpha}(x)| = 0$. Hence the system

$$x^{1-\alpha} W_{q,\alpha}(x)\Big|_{x=0} \widetilde{C} = \widetilde{0},$$

has a non zero solution $\widetilde{C}$. For the function $y(x) := \sum_{j=1}^n c_j u_j(x)$, $x \in (0,a]$, which is a solution of (2.19) in $(0,a]$ it holds that

$$x^{1-\alpha}\mathcal{D}_q^{k\alpha} y(x)\Big|_{x=0} = 0, \quad k = 0,1,\dots,n-1.$$

Therefore, by proposition 2.1, $x^{1-\alpha}y(x) = 0$ for all $x \in [0,a]$ and consequently $\left\{x^{1-\alpha}u_j(x)\right\}_{j=1}^n$ are linearly dependent in $[0,a]$, which is a contradiction. Hence (2.23) should hold. ∎

Let M be the vector space of all solutions of the homogeneous equation (2.19). We call a fundamental set of solutions of (2.19) any set of linearly independent solutions that form a basis of M.

THEOREM 2.4. *M is a vector space of dimension n.*

P r o o f. Let $\{\phi_i\}_{i=1}^n$ be n solutions of (2.19) satisfying the initial conditions

$$\lim_{x\to 0^+} x^{1-\alpha}\mathcal{D}_q^{(k-1)\alpha}\phi_j(x) = \frac{\delta_{kj}}{\Gamma_q(\alpha)}, \quad k,j = 1,\dots,n.$$

Since

$$\lim_{x\to 0^+} x^{n-n\alpha} W_{q,\alpha}(\phi_1,\dots,\phi_n)(x) = \frac{1}{\Gamma_q^n(\alpha)},$$

then, from proposition 2.3, the functions $\{\phi_i\}_{i=1}^n$ are linearly independent. Let $y \in M - \{0\}$. Then there exist constants $\{b_k\}_{k=1}^n$, not all zeros, such that $\lim_{x\to 0^+} x^{1-\alpha}\mathcal{D}_q^{(k-1)\alpha}y(x) = b_k$. Set $\widetilde{b} = (b_1, b_2, \dots, b_n)^\top$. Then the system

$$\lim_{x\to 0^+} x^{1-\alpha} W_q(\phi_1, \dots, \phi_n)(x)\widetilde{C} = \widetilde{b}$$

has a non zero solution $\widetilde{C}$, $\widetilde{C} = (c_1, c_2, \dots, c_n)$. Let $z(x) = \sum_{j=1}^n c_j\phi_j(x)$, $x \in (0, a]$. Then

$$\lim_{x\to 0^+} x^{1-\alpha}\mathcal{D}_q^{(k-1)\alpha}(y - z)(x) = 0, \quad k = 1, 2, \dots, n.$$

Consequently, from Proposition 2.1, $z(x) \equiv y(x)$, $x \in (0, a]$. Thus y is written uniquely as a linear combination of the functions $\{\phi_i\}_{i=1}^n$. Hence $\{\phi_i\}_{i=1}^n$ is a basis of M. ∎

The proofs of the following results are direct and are omitted.

Corollary 2.1. *A set $\{u_j(x)\}_{j=1}^n$ of n linearly independent solutions of* (2.19) *is a fundamental set if and only if*

$$\lim_{x\to 0^+} x^{n-n\alpha} |W_q(u_1, \dots, u_n)| \neq 0.$$

Proposition 2.5. *If y_p is a particular solution to the equation* (2.15)*, then the general solution to this equation is given by $y_g = y_h + y_p$, where y_h is the general solution to the associated homogeneous equation* (2.19).

3. Solutions of linear fractional q-difference equations with constant coefficients

In this section we are concerned with constructing a fundamental set of solutions of (2.2) when it has constant coefficients. Let

$$L_{q,n\alpha}y(x) := \sum_{k=0}^{n} a_k \mathcal{D}_q^{k\alpha} y(x) = 0, \tag{3.1}$$

where the coefficients $\{a_k\}_{k=0}^{n-1}$ are real constants and $a_n \neq 0$.

The characteristic polynomial $P(\lambda)$ of (3.1) is defined by

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n\,, \quad \lambda \in \mathbb{C}. \tag{3.2}$$

From now on λ_i, $1 \leqslant i \leqslant K$ denote the distinct roots of $P(\lambda)$ and μ_i denotes the multiplicity of λ_i, so that $\sum_{i=1}^K \mu_i = n$. As in the usual case we shall seek a solution in the form

$$y(x, \lambda) = x^{\alpha-1} e_{\alpha,\alpha}(\lambda x^\alpha; q) = x^{\alpha-1} \sum_{j=0}^{\infty} \frac{(\lambda x^\alpha)^j}{\Gamma_q(\alpha j + \alpha)}, \quad |x| < (1-q)^{-1}.$$

Since $\mathcal{D}_q^{k\alpha} y(x,\lambda) = \lambda^k y(x,\lambda)$, $k \in \mathbb{N}$, then we have

$$L_{q,n\alpha} y(x,\lambda) = P_n(\lambda) y(x,\lambda). \tag{3.3}$$

The following assertion is true for complex $\lambda \in \mathbb{C}$.

LEMMA 3.1. *If $\lambda \in \mathbb{C}$ is a root of the characteristic polynomial* (3.2), *then*

$$\frac{\partial^l}{\partial \lambda^l} \left(L_{q,n\alpha} y(x,\lambda) \right) = L_{q,n\alpha} \left(\frac{\partial^l}{\partial \lambda^l} y(x,\lambda) \right), \quad l \in \mathbb{N}, \tag{3.4}$$

and

$$\frac{\partial^l}{\partial \lambda^l} y(x,\lambda) = x^{l\alpha+\alpha-1} \sum_{m=0}^{\infty} (m+l)(m+l-1)\dots(m+1) \frac{(\lambda x^\alpha)^m}{\Gamma_q(m\alpha + l\alpha + \alpha)}.$$

P r o o f. The lemma follows from the linearity of the operators $\frac{\partial^l}{\partial \lambda^l}$ and $L_{q,n\alpha}$. ■

LEMMA 3.2. *For i; $i = 1, \dots, K$, and l, $l = 1, \dots, \mu_i$, the functions*

$$\phi_{\alpha,l}(x,\lambda_i) := x^{l\alpha+\alpha-1} \sum_{m=0}^{\infty} (m+l)(m+l-1)\dots(m+1) \frac{(\lambda_i x^\alpha)^m}{\Gamma_q(m\alpha + l\alpha + \alpha)}, \tag{3.5}$$

$|\lambda_i||x(1-q)|^\alpha < 1$, *are linearly independent solutions of* (3.1).

P r o o f. From equations (3.3)–(3.4) and the classical Leibniz rule, we obtain

$$\begin{aligned} L_{q,n\alpha}\left(\phi_{\alpha,l}(x,\lambda_i)\right) &= L_{q,n\alpha}\left(\frac{\partial^l}{\partial \lambda^l} e_{\alpha,1}(\lambda x; q)\right)\Big|_{\lambda=\lambda_i} \\ &= \sum_{r=0}^{l} \binom{l}{r} P^{(r)}(\lambda) \phi_{i,l}^{(l-r)}(x,\lambda)\Big|_{\lambda=\lambda_i} = 0, \end{aligned}$$

for $l = 0, 1, \dots, \mu_i - 1$. So the functions defined in (3.5) are solution of (3.1). The linear independence follows from Proposition 2.3 since

$$\lim_{x \to 0^+} x^{k-\alpha k} \left| W_{q,\alpha}\left(\phi_{\alpha,1}, \phi_{\alpha,2}, \dots, \phi_{\alpha,\mu_i}\right)(x,\lambda_i) \right| = \frac{1}{\Gamma_q^k(\alpha)}.$$

■

This lemma and the above discussion leads to the following theorem.

THEOREM 3.1. *The set* $\{\phi_{\alpha,r}(x,\lambda_i)\}_{r=0}^{\mu_i-1}$ *of* (3.5) *is a linearly independent set of solutions of* (3.1). *Moreover, the set*

$$\left\{\{\phi_{\alpha,r}(x,\lambda_i)\}_{r=0}^{m_i-1}\,,\ i=1..k\right\}$$

is a fundamental set of solutions of (3.1).

EXAMPLE 3.1. Consider the fractional differential equation

$$\mathcal{D}_q^{2\alpha}y(x)-y(x)=0.$$

The characteristic polynomial $P(\lambda)=\lambda^2-1$ has two distinct roots $\lambda_1=1$, $\lambda_2=-1$. Hence the general solution is given by

$$y(x)=c_1x^{\alpha-1}e_{\alpha,1}(x;q)+c_2x^{\alpha-1}e_{\alpha,2}(-x;q),$$

where c_1 and c_2 are arbitrary constants.

4. General solution in the non homogenous case

In this section we seek a general solution for the non-homogenous equation

$$L_{q,n\alpha}y(x)=f(x). \tag{4.1}$$

by applying the Laplace transform method to derive a particular solution $y_p(x)$ of (4.1).

In [14], Hahn defined two q-analogs of the Laplace transform. We are interested in the one defined by

$${}_qL_sf(x)=\phi(s)=\frac{1}{1-q}\int_0^{s^{-1}}(-qsx;q)_\infty\,f(x)\,d_qx. \tag{4.2}$$

As an example, if

$$\phi_r(x)=\frac{x^r}{\Gamma_q(r+1)},\quad r>-1,\quad \text{then}\quad {}_qL_s\phi_r(x)=\frac{(1-q)^r}{s^{r+1}}. \tag{4.3}$$

Abdi in [1] studied certain properties of these q-transforms. In [2] he used these analogs to solve linear q-difference equations with constant coefficients and certain allied equations. In [9], Annaby and Mansour used (4.2) to construct a fundamental set of solutions for a certain linear R-L fractional q-difference equation with constant coefficients. Ismail in [15] defined the convolution of two functions F, G to be

$$(F * G) = \frac{1}{1-q}\int_0^x F(t)\varepsilon^{-qt}G(x)\, d_qt, \tag{4.4}$$

where ε^y is defined by Ismail in [15]. It is proved by Hahn, cf. [14] that

$${}_qL_s(F * G) = {}_qL_sF\, {}_qL_sG. \tag{4.5}$$

The q-Laplace transforms of the R-L fractional q-integral and q-derivative are given in the following lemma, cf. [9]

LEMMA 4.1. *If $F \in \mathcal{L}_q^1[0,a]$ and $\Phi(s) := {}_qL_sF(x)$, then*

$${}_qL_sI_q^\beta F(x) = \frac{(1-q)^\beta}{s^\beta}\Phi(s), \quad \beta > 0. \tag{4.6}$$

If $0 < \beta < 1$ and $I_q^{1-\beta}F(x) \in \mathcal{AC}_q[0,a]$, then

$${}_qL_sD_q^\beta F(x) = \frac{s^\beta}{(1-q)^\beta}\Phi(s) - I_q^{1-\beta}F(0). \tag{4.7}$$

LEMMA 4.2. *The q-Laplace transform of the Riemann–Liouville sequential q-derivative of order $m\alpha$ is given by*

$${}_qL_s\mathcal{D}_q^{m\alpha}y(x) = p^{m\alpha}\phi(s) - \sum_{l=0}^{m-1} p^{\alpha(m-1-l)}I_q^{1-\alpha}\mathcal{D}_q^{\alpha l}, \quad p = \frac{s}{1-q}.$$

P r o o f. The proof of this lemma follows by induction on m and by using (4.7). ∎

LEMMA 4.3. *For each i, $i = 1,\dots,k$, $l = 1,2,\dots,\mu_i$,*

$${}_qL_s\left(\phi_{\alpha,l}(x,\lambda_i)\right) = \frac{l!}{1-q}(p^\alpha - \lambda_i)^{-l-1}, \quad |p|^\alpha > |\lambda_i| \tag{4.8}$$

is valid in the disk $\{x \in \mathbb{R} : |\lambda_i||x(1-q)|^\alpha < 1\}$, where $p := \dfrac{s}{1-q}$.

P r o o f. From the properties of the q-Laplace transform, cf. [2], (3.5) and (4.3), we obtain

$$\begin{aligned}
{}_qL_s\phi_{\alpha,l}(x,\lambda_i) &= {}_qL_s\sum_{m=0}^\infty (m+l)(m+l-1)\dots(m+1)\frac{\lambda_i^m x^{m\alpha+l\alpha+\alpha-1}}{\Gamma_q(m\alpha+l\alpha+\alpha)} \\
&= \frac{p^{-l\alpha-\alpha}}{1-q}\sum_{m=0}^\infty (m+l)(m+l-1)\dots(m+1)(\lambda_i\, p^{-\alpha})^m.
\end{aligned}$$

So for $|\lambda_i| < p^\alpha$ we get

$$\begin{aligned} {}_qL_s\phi_{\alpha,l}(x,\lambda_i) &= \frac{p^{-l\alpha-\alpha}}{1-q}\frac{d^l}{dz^l}\sum_{m=0}^{\infty} z^m\Big|_{z=\lambda_i p^{-\alpha}} = \frac{p^{-l\alpha-\alpha}}{(1-q)}\frac{d^l}{dz^l}\frac{1}{1-z}\Big|_{z=\lambda_i p^{-\alpha}} \\ &= \frac{l!}{(1-q)(p^\alpha-\lambda_i)^{l+1}}. \end{aligned}$$

■

THEOREM 4.1. *Let $\{\lambda_j\}_{j=1}^K$ be the K distinct roots of the multiplicity $\{\mu_j\}_{j=1}^K$ of the characteristic polynomial $P_n(\lambda)$ associated with the homogeneous equation $L_{q,n\alpha}y(x) = 0$. Let $Q_{n-1}(\lambda)$ be the polynomial defined by*

$$Q_{n-1}(\lambda) = \sum_{l=0}^{n-1} d_l\lambda^l, \quad d_l := \sum_{k=0}^{n-l-1} a_k I_q^{1-\alpha}\mathcal{D}_q^{k\alpha}y(0).$$

Let $\{\gamma_{j,r}\}$, $\{\delta_{j,r}\}$,$j = 1,\ldots,k,\ r = 1,\ldots,\mu_j$ be the constants satisfying the identities

$$\frac{1}{P_n(\lambda)} = \sum_{j=1}^{K}\sum_{r=1}^{\mu_j}\frac{\delta_{j,r}}{(\lambda-\lambda_j)^r}, \quad \frac{Q_{n-1}(p)}{P_n(p)} = \sum_{j=1}^{K}\sum_{r=1}^{\mu_j}\frac{\gamma_{j,r}}{(\lambda-\lambda_j)^r}. \tag{4.9}$$

Then the general solution of (4.1) *is given by*

$$y(x) = \sum_{j=1}^{K}\sum_{r=1}^{\mu_j}\frac{(1-q)}{r-1!}\Big(\gamma_{j,r}\phi_{\alpha,r}(x,\lambda_j) + \delta_{j,r}f(x)*\phi_{\alpha,r}(x,\lambda_j)\Big). \tag{4.10}$$

P r o o f. First, it should be noted that the constants $\{\gamma_{j,r}\}$, $\{\delta_{j,r}\}$, $j = 1,\ldots,k,\ r = 1,\ldots,\mu_j$ are uniquely determined by applying the method of partial fraction on the fractional functions $\dfrac{1}{P_n(\lambda)}$ and $\dfrac{Q_{n-1}(\lambda)}{P_n(\lambda)}$ respectively. Applying the q-Laplace transform on the two sides of (4.1) gives

$$\left(\sum_{r=0}^{n} a_r p^{r\alpha}\right)\phi(s) - \sum_{m=1}^{n}\sum_{l=0}^{m-1} a_{m-1-l}\left(I_q^{1-\alpha}\mathcal{D}_q^{\alpha l}y(0)\right)p^{l\alpha} = F(s),\ p = \frac{s}{1-q}.$$

Since $\sum_{m=1}^{n}\sum_{l=0}^{m-1} a_{m-1-l}\left(I_q^{1-\alpha}\mathcal{D}_q^{\alpha l}y(0)\right)p^{l\alpha} =$

$$\sum_{l=0}^{n-1}\left(\sum_{m=l+1}^{n} a_{m-1-l}\left(I_q^{1-\alpha}\mathcal{D}_q^{\alpha(m-l-1)}y(0)\right)\right)p^{l\alpha} = \sum_{l=0}^{n-1} d_l\,p^{l\alpha} = Q_{n-1}(p^\alpha),$$

then $\phi(s) = \dfrac{Q_{n-1}(p^\alpha)}{P_n(p^\alpha)} + \dfrac{F(s)}{P_n(p^\alpha)}$. From (4.9) and (4.5) we obtain

$$\phi(s) = \sum_{j=1}^{k}\sum_{r=1}^{\mu_j} \frac{\gamma_{j,r}}{(p^\alpha - \lambda_j)^r} + \sum_{j=1}^{K}\sum_{r=1}^{\mu_j} \frac{\delta_{j,r}}{(p^\alpha - \lambda_j)^r} F(s).$$

$$\text{Thus,}\quad y(x) = \sum_{j=1}^{K}\sum_{r=1}^{\mu_j} {}_qL_s^{-1}\left(\frac{\gamma_{j,r}}{(p^\alpha - \lambda_j)^r} + F(s)\frac{\delta_{j,r}}{(p^\alpha - \lambda_j)^r}\right). \tag{4.11}$$

From (4.8) and (4.5) we get, for $|p^\alpha| > \max_{1\leqslant j\leqslant K} |\lambda_j|$,

$$\frac{1}{(p^\alpha - \lambda_j)^r} = {}_qL_s\left(\frac{(1-q)}{r-1!}\phi_{\alpha,r}(x,\lambda_j)\right), \tag{4.12}$$

$${}_qL_s^{-1}\frac{F(s)}{(p^\alpha - \lambda_j)^r} = f(x) * \phi_{\alpha,r}(x,\lambda_j). \tag{4.13}$$

Then substituting from (4.12) and (4.13) in (4.10) gives (4.10) and completing the proof. ∎

Example 4.1. Consider the sequential fractional q-difference equation of order 2α

$$\mathcal{D}_q^{2\alpha} y(x) - \mathcal{D}_q^{\alpha} y(x) - 2y(x) = f(x). \tag{4.14}$$

The characteristic polynomial $P_2(\lambda)$ of (4.14) is given by

$$P_2(\lambda) = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

It has two distinct roots $\lambda_1 = -1$ and $\lambda_2 = 2$. Therefore the general solution of the homogeneous equation (4.14) is given by

$$y(x) = c_1 x^{\alpha-1} e_{\alpha,\alpha}(-x^\alpha; q) + c_2 x^{\alpha-1} e_{\alpha,\alpha}(2x^\alpha; q),$$

where c_1 and c_2 are arbitrary constants. One can verify that

$$\gamma_{1,1} = \frac{1}{3}(d_1 - d_0),\ \gamma_{2,1} = \frac{1}{3}(2d_1 - d_0),\ \delta_{1,1} = -\frac{1}{3},\ \text{and}\ \delta_{2,1} = \frac{1}{3}.$$

That is,

$$y_p(x) = \gamma_{1,1}\phi_{\alpha,1}(x,-1) + \gamma_{2,1}\phi_{\alpha,1}(x,2) + \frac{1}{3} f(x) * (\phi_{\alpha,1}(x,2) - \phi_{\alpha,1}(x,-1)).$$

Now we compute y_p in case of $f(x) = x$. Set $g(x) := \phi_{\alpha,1}(x,2) - \phi_{\alpha,1}(x,-1)$. Then

$$f(x) * g(x) = \frac{1}{1-q}\int_0^x g(t)\varepsilon^{-qt} x\, d_qt = \frac{1}{1-q}\int_0^x g(t)(x - qt)\, d_qt. \tag{4.15}$$

Applying the q-integration by part rule (1.5) with

$$a = 0,\ b = x,\ \ u(t) = x - t, \quad \text{and} \quad D_q v(t) = g(t)$$

gives

$$\begin{aligned} f(x) * g(x) &= \frac{1}{1-q} \sum_{m=0}^{\infty} (m+1) \left(2^m + (-1)^{m+1}\right) \frac{x^{m\alpha+2\alpha}}{\Gamma_q(m\alpha + 2\alpha + 1)} \\ &= \frac{x^{2\alpha+1}}{1-q} \left(e^{(1)}_{\alpha,\alpha+2}(2x^\alpha; q) - e^{(1)}_{\alpha,\alpha+2}(-x^\alpha; q) \right), \end{aligned}$$

where by $f^{(k)}(z)$, $k \in \mathbb{Z}^+$, we mean $\dfrac{d^k}{dz^k} f(z)$. Thus

$$\begin{aligned} y_p(x) &= \gamma_{1,1}\phi_{\alpha,1}(x, -1) + \gamma_{2,1}\phi_{\alpha,1}(x, 2) \\ &+ \frac{x^{2\alpha+1}}{3(1-q)} \left(e^{(1)}_{\alpha,\alpha+2}(2x^\alpha; q) - e^{(1)}_{\alpha,\alpha+2}(-x^\alpha; q) \right). \end{aligned}$$

Remark 4.1. We would like to mention that the results of this paper hold if we consider linear sequential q-difference equations where the fractional q-derivative is the Caputo q-derivative, cf. [9, 22]. The Caputo q-derivative of order α is defined as

$$^{c}D_q^{\alpha} f(x) := I_q^{k-\alpha} D_q^k f(x), \quad k = [\alpha] + 1,\ f \in \mathcal{AC}_q^{(k)}[0, a]. \tag{4.16}$$

References

[1] W.H. Abdi, On q-Laplace transforms. *Proc. Nat. Acad. Sci. India* **29 (A)** (1961), 389-408.

[2] W.H. Abdi, On certain q-difference equations and q-Laplace transforms. *Proc. Nat. Acad. Sci. India* **28** (1962), 1-15.

[3] M.H. Abu-Risha, M.H. Annaby, M.E.H. Ismail, and Z.S. Mansour, Linear q-difference equations. *Z. Anal. Anwend.* **26** (2007), 481-494.

[4] M.H. Abu Risha, M.H. Annaby, M.E.H. Ismail, and Z.S Mansour, Existence and uniqueness theorems of q-difference equations, Submitted.

[5] R.P. Agarwal, Certain fractional q-integrals and q-derivatives. *Proc. Camb. Phil. Soc.* **66** (1969), 365-370.

[6] W.A. Al-Salam, Some fractional q-integrals and q-derivatives. *Proc. Edinburgh Math. Soc. (2)* **15** (1966/1967), 135-140.

[7] W.A. Al-Salam, q-analogue of the Cauchy formula. *Proc. Amer. Math. Soc.* **17** (1966), 616-621.

[8] G.E. Andrews, R. Askey, and R. Roy, *Special Functions.* Cambridge University Press, Cambridge (1999).

[9] M.H. Annaby and Z.S. Mansour, q-fractional calculus and equations. Submitted.

[10] N.M. Atakishiyev, On a one-parameter family of q-exponential functions. *J. Phys. A: Math. Gen.* **29** (1996), L223-L227.

[11] G. Gasper and M. Rahman, *Basic Hypergeometric Series.* Cambridge university Press, Cambridge (2004).

[12] R. Gorenflo, A.A. Kilbas, and S.B. Yakubovich, On Abel type integral equations of first kind. *J. Fract. Calc.* **5** (1994), 59-68.

[13] R. Gorenflo and F. Mainardi, Fractional calculus: Integral and differential equations of fractional order. In: A. Carpinteri and F. Mainardi (Editors), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien (1997), 223-276.

[14] W. Hahn, Beiträge zur Theorie der Heineschen Reihen. *Math. Nachr.* **2** (1949), 340-379.

[15] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable.* Cambridge University Press, Cambridge (2005).

[16] F.H. Jackson, A basic-sine and cosine with symbolical solutions of certain differential equations. *Proc. Edinburgh Math. Soc.* **22** (1903-1904), 28-34.

[17] F.H. Jackson, A generalization of the function $\Gamma(n)$ and x^n. *Proc. Roy. Soc. London* **74** (1904), 64-72.

[18] F.H. Jackson, On q-definite integrals. *Quart. J. Pure and Appl. Math.* **41** (1910), 193-203.

[19] S.L. Kalla, R.K. Yadav and S.D. Purohit, On The Riemann-Liouville fractional q-integral operator involving a basic analog of Fox H-function. *Fract. Calc. Appl. Anal.* **9**, No 3 (2006), 313-322.

[20] A.A. Kilbas, H.M. Srivastava, and J.I. Trujillo, *Theory and Applications of Fractional Differential Equations.* Elsevier, London (2006).

[21] S.D. Purohit and S.L. Kalla, On q-Laplace transforms of the q-Bessel functions. *Fract. Calc. Appl. Anal.* **10**, No 2 (2007), 189-196.

[22] P.M. Rajković, S.D. Marinković, and M.S. Stanković, On q-analogues of Caputo derivatives and Mittag-Leffler function. *Fract. Calc. Appl. Anal.* **10**, No 4 (2007), 359-374.

[23] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives. Theory and Applications.* Gordon and Breach Science Publisher, Switzerland - USA (1993).

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